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Robust H_∞ Control Synthesis Method and Its Application to Benchmark Problems

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This paper presents a robust H_∞ control synthesis method for structured parameter uncertainty. The robust H_∞ control design methodology is also incorporated with the so-called internal model principle for persistent-disturbance rejection. A noncollocated control problem of flexible space structures subject to parameter variations is used to illustrate the design methodology. It is shown that the proposed design method invariably makes use of nonminimum-phase compensation and that it achieves the desired asymptotic disturbance rejection by having a disturbance rejection "dipole."

I. Introduction

IN this paper we present a robust control design methodology and its application to the benchmark problem of Ref. 1. The benchmark problems are concerned with robust control design for a simple yet meaningful two-mass-spring system first introduced in Ref. 2. The control design methodology is based on the recent advances in H_∞ control[3,4] and an approach developed in Refs. 5–7 for controller robustification with respect to structured parameter uncertainty. It further incorporates the internal model principle for robust asymptotic disturbance rejection in the presence of persistent external disturbances as well as structured parameter uncertainty.

The robust servomechanism problem for asymptotic tracking and disturbance rejection has been investigated by many researchers over the last three decades, and it seems that the internal model principle is the most intuitive solution to such a control problem. A detailed discussion and a comprehensive list of references for this subject can be found in Ref. 8. In this paper, the internal model principle is incorporated for the first time with the robust H_∞ control synthesis methodology of Refs. 5–7 and is applied to the benchmark problems of Ref. 1.

A linear-quadratic-regulator design incorporating the internal model principle was first investigated in Ref. 9 for the Space Station Freedom excited by persistent aerodynamic disturbances. In Refs. 5 and 7, a robust H_∞ control synthesis technique is developed for a case with nonlinear multiparameter variations and is applied to the Space Station control problem, resulting in a significant improvement in parameter robustness margins. The concept of a disturbance rejection "dipole" is introduced in Ref. 10 from a classical control viewpoint and has been experimentally validated by ground test of the Mini-Mast flexible structure.[11]

This paper will show that the proposed design method invariably makes use of nonminimum-phase compensation for a class of noncollocated control problems with structured parameter uncertainty and that it achieves the desired asymptotic disturbance rejection by having a disturbance rejection dipole.

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The remainder of this paper is organized as follows. In Sec. II, the benchmark problem is briefly reviewed. In Sec. III, a robust H_∞ control synthesis methodology incorporating structured parameter variations is presented. Robust control designs for problems 1 and 2 of the benchmark problems are presented in Secs. IV and V, respectively. The ∞-norm parameter margin is employed in Sec. V as a stability robustness measure with respect to multiparameter variations. In Sec. VI, the internal model principle is incorporated with the robust H_∞ control synthesis methodology and is applied to problem 3 of the benchmark problems in the presence of persistent excitation as well as structured parameter variations.

II. Benchmark Problems

Consider a two-mass-spring system shown in Fig. 1, which is a generic model of an uncertain dynamical system with a rigid-body mode and one vibration mode.[1,2] It is assumed that for the nominal system $m_1 = m_2 = 1$ and $k = 1$ with appropriate units and time is in units of seconds. A control force acts on body 1, and the position of body 2 is measured, resulting in a noncollocated control problem.

This system can be represented in state-space form as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m_1 & k/m_1 & 0 & 0 \\ k/m_2 & -k/m_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/m_1 \\ 0 \end{bmatrix} (u + w_1) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m_2 \end{bmatrix} w_2 \tag{1a}$$

$$y = x_2 + v \tag{1b}$$

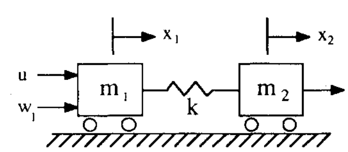

Fig. 1 Two-mass-spring system.

$$z = x_2 \tag{1c}$$

where x_1 and x_2 are the positions of body 1 and body 2, respectively; x_3 and x_4 the velocities of body 1 and body 2, respectively; u the control input acting on body 1; y the measured output; w_1 and w_2 the plant disturbances acting on body 1 and body 2, respectively; v the sensor noise; and z the output to be controlled.

The transfer function description of the plant is

$$y(s) = \frac{k}{s^2[m_1 m_2 s^2 + k(m_1 + m_2)]} u(s) \tag{2}$$

Constant-gain linear feedback controllers are to be designed for the following three different problems:

Problem 1

For a unit impulse disturbance exerted on body 1 and/or body 2, the controlled output ($z = x_2$) must have a settling time of about 15 s for the nominal system with $m_1 = m_2 = k = 1$. The closed-loop system should be stable for $0.5 \le k \le 2.0$ and $m_1 = m_2 = 1$. The sensor noise, actuator saturation, and high-frequency rolloff must be considered to reflect practical control design tradeoffs.

Problem 2

Maximize a stability robustness measure with respect to the three uncertain parameters m_1, m_2, and k whose nominal values are $m_1 = m_2 = k = 1$. For a unit impulse disturbance exerted on body 1 and/or body 2, the controlled output must have a settling time of about 15 s for the nominal system with $m_1 = m_2 = k = 1$.

Problem 3

There is a sinusoidal disturbance with known frequency of 0.5 rad/s acting on body 1 and/or body 2, but whose amplitude and phase, although constant, are not available to the designer. The closed-loop system must achieve asymptotic disturbance rejection for the controlled output with a 20-s settling time for $m_1 = m_2 = 1$ and $0.5 \le k \le 2.0$.

III. Robust H_∞ Control Synthesis Methodology

In Refs. 5-7, a robust control synthesis technique is investigated by converting the parameter-insensitive control design problem into a conventional H_∞ control problem. An interesting feature of the technique explored in Refs. 5-7 is the fact that the state-space solution to a standard H_∞ control problem, developed in Refs. 3 and 4, can be simply utilized by redefining the structured parameter variations in terms of fictitious inputs and outputs. In this section, such a robust H_∞ control synthesis technique for structured parametric uncertainty is presented as a brief, self-contained summary of the methodology.

The H_∞-norm of a real-rational matrix $T(s)$ is defined as

$$\begin{aligned} \|T(s)\|_\infty &\triangleq \sup\{\|T(s)\| : \mathrm{Re}(s) > 0\} \\ &= \sup_\omega \|T(j\omega)\| \\ &= \sup_\omega \bar{\sigma}[T(j\omega)] \end{aligned}$$

where $\bar{\sigma}[T(j\omega)]$ is defined as the largest singular value of $T(j\omega)$ for a given ω. The H_∞ space consists of functions that are stable and bounded.

Consider a linear, time-invariant system described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t) \\ z(t) &= C_1 x(t) + D_{11} w(t) + D_{12} u(t) \\ y(t) &= C_2 x(t) + D_{21} w(t) + D_{22} u(t) \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$, $w \in \mathbb{R}^{m_1}$, $u \in \mathbb{R}^{m_2}$, $z \in \mathbb{R}^{p_1}$, and $y \in \mathbb{R}^{p_2}$ are the state, disturbance input, control input, controlled output, and measured output vectors, respectively.

The transfer function representation of this system is given by

$$\begin{bmatrix} z(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix} \tag{4}$$

To utilize the concept of an internal feedback loop, the system with uncertain parameters is described as

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \begin{bmatrix} \hat{A} & \hat{B}_1 & \hat{B}_2 \\ C_1 & D_{11} & D_{12} \\ \hat{C}_2 & \hat{D}_{21} & \hat{D}_{22} \end{bmatrix} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{5}$$

where C_1, D_{11}, and D_{12} are not subject to parameter variations. The perturbed system matrix in Eq. (5) can be linearly decomposed as follows:

$$\begin{bmatrix} \dot{x} \\ z \\ y \end{bmatrix} = \left\{ \begin{bmatrix} A & B_1 & B_2 \\ C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{bmatrix} + \mathcal{E} \right\} \begin{bmatrix} x \\ w \\ u \end{bmatrix} \tag{6}$$

where the first matrix on the right-hand side is the nominal system matrix and $\mathcal{E}$ is the perturbation matrix defined as

$$\mathcal{E} \triangleq \begin{bmatrix} \Delta A & \Delta B_1 & \Delta B_2 \\ 0 & 0 & 0 \\ \Delta C_2 & \Delta D_{21} & \Delta D_{22} \end{bmatrix} \tag{7}$$

Suppose that there are m independent uncertain parameters p_i and that the perturbation matrix $\mathcal{E}$ is decomposed as

$$\mathcal{E} = -\begin{bmatrix} M_x \\ 0 \\ M_y \end{bmatrix} E \begin{bmatrix} N_x & N_w & N_u \end{bmatrix} = -MEN \tag{8}$$

where

$$E = \mathrm{diag}\{\Delta p_1, \ldots, \Delta p_m\} \tag{9}$$

It should be noted that the decomposition of the perturbation matrix is not unique.

By introducing the following new variables

$$\tilde{z} \triangleq \begin{bmatrix} N_x & 0 & N_w & N_u \end{bmatrix} \begin{bmatrix} x \\ \tilde{w} \\ w \\ u \end{bmatrix} \tag{10a}$$

$$\tilde{w} \triangleq -E\tilde{z} \tag{10b}$$

where $\tilde{w}$ is the fictitious input; $\tilde{z}$ the fictitious output; and E the fictitious, internal feedback loop gain matrix, we get the following system description:

$$\begin{bmatrix} \dot{x} \\ \tilde{z} \\ z \\ y \end{bmatrix} = \begin{bmatrix} A & M_x & B_1 & B_2 \\ N_x & 0 & N_w & N_u \\ C_1 & 0 & D_{11} & D_{12} \\ C_2 & M_y & D_{21} & D_{22} \end{bmatrix} \begin{bmatrix} x \\ \tilde{w} \\ w \\ u \end{bmatrix} \tag{11a}$$

$$\tilde{w} = -E\tilde{z} \tag{11b}$$

The overall system can then be represented in transfer function form as

$$\begin{bmatrix} \tilde{z} \\ z \\ y \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & G_{13} \\ G_{21} & G_{22} & G_{23} \\ G_{31} & G_{32} & G_{33} \end{bmatrix} \begin{bmatrix} \tilde{w} \\ w \\ u \end{bmatrix} \tag{12a}$$

$$\tilde{w} = -E\tilde{z} \tag{12b}$$

$$u = -K(s)y \tag{12c}$$

where $K(s)$ is a feedback compensator to be designed.

To utilize the state-space representation given by Eq. (3), $\hat{z}$, $\hat{w}$, and the associated matrices are defined as follows:

$$\hat{z} \leftarrow \begin{bmatrix} \tilde{z} \\ z \end{bmatrix}, \qquad \hat{w} \leftarrow \begin{bmatrix} \tilde{w} \\ w \end{bmatrix}$$

$$B_1 \leftarrow \begin{bmatrix} M_x & B_1 \end{bmatrix}, \qquad C_1 \leftarrow \begin{bmatrix} N_x \\ C_1 \end{bmatrix}$$

$$D_{11} \leftarrow \begin{bmatrix} 0 & N_w \\ 0 & D_{11} \end{bmatrix}, \qquad D_{12} \leftarrow \begin{bmatrix} N_u \\ D_{12} \end{bmatrix}$$

$$D_{21} \leftarrow \begin{bmatrix} M_y & D_{21} \end{bmatrix} \tag{13}$$

The resulting modified state-space representation of the system is then given by

$$\dot{x}(t) = Ax(t) + B_1\hat{w}(t) + B_2u(t)$$

$$\hat{z}(t) = C_1x(t) + D_{11}\hat{w}(t) + D_{12}u(t)$$

$$y(t) = C_2x(t) + D_{21}\hat{w}(t) + D_{22}u(t) \tag{14}$$

After closing the control loop with a stabilizing controller $K(s)$, we get the following representation of the closed-loop system (but with the internal feedback loop broken):

$$\hat{z} = T_{\hat{z}\hat{w}}\hat{w} \tag{15}$$

where

$$T_{\hat{z}\hat{w}} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \tag{16a}$$

$$T_{11} = G_{11} - G_{13}K(I + G_{33}K)^{-1}G_{31} \tag{16b}$$

$$T_{12} = G_{12} - G_{13}K(I + G_{33}K)^{-1}G_{32} \tag{16c}$$

$$T_{21} = G_{21} - G_{23}K(I + G_{33}K)^{-1}G_{31} \tag{16d}$$

$$T_{22} = G_{22} - G_{23}K(I + G_{33}K)^{-1}G_{32} \tag{16e}$$

The actual closed-loop transfer matrix from w to z under plant perturbations is represented as

$$T_{zw} = T_{22} - T_{21}E(I + T_{11}E)^{-1}T_{12} \tag{17}$$

The following two theorems provide sufficient conditions for stability/performance robustness.

Theorem 1 (Stability Robustness)

If $\|T_{11}(s)\|_\infty < \gamma$, then $T_{zw}(s,\alpha E)$ $\forall\alpha \in [0,1]$ is stable for $\|E\| \le \gamma^{-1}$.

Theorem 2 (Performance Robustness)

If $\|T_{\hat{z}\hat{w}}\|_\infty < \gamma$, then $T_{zw}(s,\alpha E)$ $\forall\alpha \in [0,1]$ is stable and $\|T_{zw}(s,\alpha E)\|_\infty < \gamma$ $\forall\alpha \in [0,1]$ with $\|E\| \le \gamma^{-1}$.

The following theorem[4,5] gives a robust H_∞-suboptimal controller that satisfies the condition in theorem 2.

Theorem 3 (H_∞-Suboptimal Controller)

Assume that

1) (A, B_2) is stabilizable and (C_2, A) is detectable.
2) $D_{12}^T[C_1 \quad D_{12}] = [0 \quad I]$.
3) $\begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} D_{21}^T = \begin{bmatrix} 0 \\ I \end{bmatrix}$.
4) The rank of $P_{12}(j\omega)$ and $P_{21}(j\omega)$ is m_2 and p_2, respectively, for all ω.
5) $D_{11} = 0$ and $D_{22} = 0$.

There exists an internally stabilizing controller such that $\|T_{\hat{z}\hat{w}}\|_\infty < \gamma$, if and only if the following Riccati equations

$$0 = A^TX + XA - X(B_2B_2^T - \gamma^{-2}B_1B_1^T)X + C_1^TC_1 \tag{18}$$

$$0 = AY + YA^T - Y(C_2^TC_2 - \gamma^{-2}C_1^TC_1)Y + B_1B_1^T \tag{19}$$

have symmetric positive semidefinite solutions X and Y such that

1) $A - (B_2B_2^T - \gamma^{-2}B_1B_1^T)X$ is stable.
2) $A - Y(C_2^TC_2 - \gamma^{-2}C_1^TC_1)$ is stable.
3) $(I - \gamma^{-2}YX)^{-1}Y$ is positive semidefinite.

An H_∞-suboptimal controller that satisfies $\|T_{zw}\|_\infty < \gamma$, where γ is a design tradeoff variable specifying an upper bound of the perturbed closed-loop transfer matrix T_{zw}, is then obtained as

$$\dot{\hat{x}} = A_c\hat{x} + Ly \tag{20a}$$

$$u = -K\hat{x} \tag{20b}$$

where

$$K = B_2^TX \tag{21}$$

$$L = (I - \gamma^{-2}YX)^{-1}YC_2^T \tag{22}$$

$$A_c = A + \gamma^{-2}B_1B_1^TX - B_2K - LC_2 \tag{23}$$

and $\hat{x}$ represents the controller state vector.

The closed-loop system (neglecting all the external inputs) is then described as

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & -B_2K \\ LC_2 & A_c \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{24}$$

It can be shown that the separation principle of the conventional linear-quadratic-Gaussian (LQG) synthesis technique does not hold in this case.

IV. Robust Control Design for Problem 1

A robust H_∞ control design for problem 1 was first presented in Ref. 6, but it is included here for the completeness of the benchmark problems and their solutions. For the state-space description of a two-mass-spring model with one uncertain parameter k, described by Eq. (1), the variation ΔA is decomposed as

$$\Delta A = -\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \Delta k \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix} \tag{25}$$

The other elements of the perturbation matrix in Eq. (7) are all zeros. Note that ΔA is spanned by the matrices

$$M_x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \qquad N_x = \begin{bmatrix} 1 & -1 & 0 & 0 \end{bmatrix}$$

where M_x is the fictitious disturbance distribution matrix spanning the columns of ΔA, and N_x is the fictitious controlled output distribution matrix spanning the rows of ΔA. The fictitious input and output for this example are then expressed as

$$\tilde{z} = Nx = x_1 - x_2, \qquad \tilde{w} = -\Delta k \tilde{z} \tag{26}$$

That is, the internal feedback loop-gain matrix E is simply the spring stiffness perturbation Δk here.

Equation (26) replaces the parameter variation in Eqs. (1), resulting in the following equations:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} (u + w_1)$$

$$+ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} w_2 + \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \tilde{w} \tag{27a}$$

$$\begin{bmatrix} \tilde{z} \\ z \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ x_2 \\ u \end{bmatrix} \tag{27b}$$

$$y = x_2 + v \tag{27c}$$

As can be seen in Eq. (27b), the control input u has been included in the controlled output as $z = [x_2 \;\; u]^T$ to minimize the control effort, too. For design simplicity, external disturbance w_2 will be assumed to be zero, but closed-loop simulation results with w_2 will be presented later.

By defining

$$\hat{w} \leftarrow \begin{bmatrix} \tilde{w} \\ w_1 \\ v \end{bmatrix}, \qquad \hat{z} \leftarrow \begin{bmatrix} \tilde{z} \\ z \end{bmatrix} \tag{28}$$

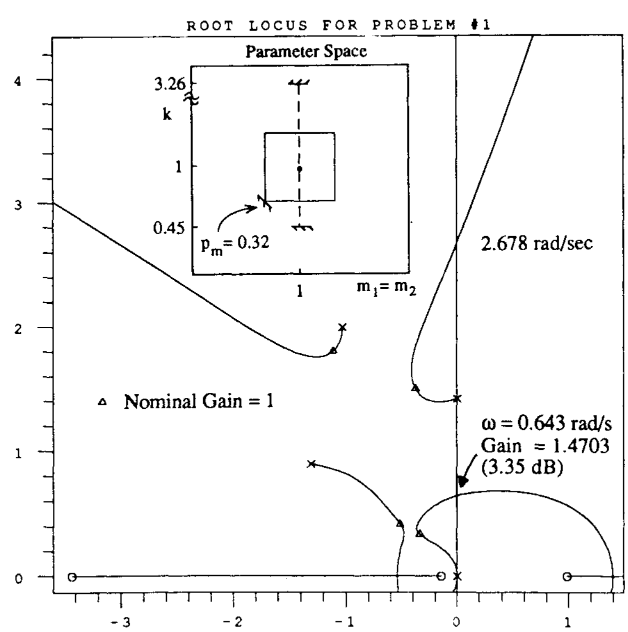

Fig. 2 Root-locus vs overall loop gain (problem 1).

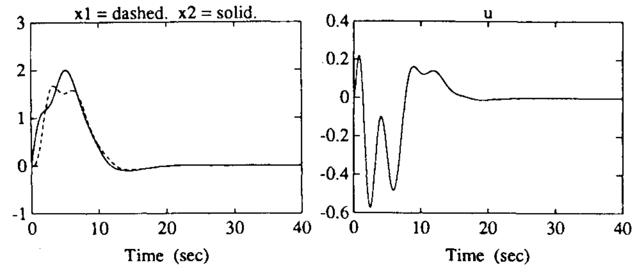

Fig. 3 Time responses to a unit-impulse disturbance acting on body 2 (problem 1).

we get the modified system matrices in Eq. (14) as

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} W, \qquad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} W$$

$$D_{11} = 0, \qquad D_{22} = 0$$

where W is a diagonal weighting matrix for $\hat{w}$.

The weighting matrix W and γ are design tradeoff variables, which are often chosen after some trial-and-error iterations. By selecting $\gamma = 1$ and the disturbance weighting matrix

$$W = \text{diag}\{0.1, 0.025, 0.025\}$$

we obtain a controller as

$$\dot{\hat{x}}_c = A_c \hat{x}_c + Ly$$

$$u = -Kx_c$$

where

$$A_c = \begin{bmatrix} 0.0 & -0.7195 & 1.0 & 0.0 \\ 0.0 & -2.9732 & 0.0 & 1.0 \\ -2.5133 & 4.8548 & -1.7287 & -0.9616 \\ 1.0063 & -5.4097 & -0.0081 & 0.0304 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.720 & 2.973 & -3.370 & 4.419 \end{bmatrix}^T$$

$$K = \begin{bmatrix} 1.506 & -0.494 & 1.738 & 0.932 \end{bmatrix}$$

or in transfer function form as

$$u(s) = \frac{-0.0827(s/0.145 + 1)(-s/0.984 + 1)}{[(s/1.586)^2 + 2(0.825)s/1.586 + 1]} \times \frac{(s/3.434 + 1)}{[(s/2.24)^2 + 2(0.459)s/2.24 + 1]} y(s) \tag{29}$$

which is a nonminimum-phase compensator. Some general comments on the merits and drawbacks of such nonminimum-phase compensation will be made later in Sec. VII. Detailed discussion on nonminimum-phase compensation can be found in Refs. 6 and 12.

Figure 2 shows a closed-loop root locus plot vs overall loop gain of this compensator. The nominal system has a gain margin of 3.35 dB and a phase margin of 24.45 deg. Such gain and phase margins can be considered to be relatively small compared with typical 6 dB and 40 deg margins employed in practice. It is, however, important to note that the control design problem here is concerned with controller robustification with respect to plant vibration pole uncertainty caused by parametric variation of the spring stiffness k. Some further discussion on this issue will be given in Sec. VII.

The nominal closed-loop poles are

$$\begin{array}{ll} -0.337 \pm 0.336j, & -0.514 \pm 0.414j \\ -0.376 \pm 1.495j, & -1.109 \pm 1.797j \end{array} \tag{30}$$

The settling time of the nominal closed-loop system can be estimated as $4/0.337 = 11.87$ s, based on the time constant of the closed-loop rigid-body mode pole. Figure 3 shows that the nominal closed-loop system settles down within 15 s, as specified, for a unit impulse exerted on body 2.

It can be easily checked that the closed-loop system is stable for $0.449 \leq k \leq 3.264$, which corresponds to the parameter perturbation range of

$$-0.551 \leq \Delta k \leq +2.264$$

It is interesting to note that the closed-loop system has a parameter margin of 0.32 with respect to three uncertain parameters m_1, m_2, and k as illustrated in Fig. 2, which will be discussed in the next section.

V. Robust Control Design for Problem 2

A "hypercube" in the space of the plant parameters, centered at a nominal point, is used as a stability robustness measure for problem 2. The robust control synthesis problem is then to find a controller that yields the largest hypercube that will fit within the existing, but unknown, region of stability in the plant's parameter space.

It is shown in Ref. 13 that, for mass-spring dynamical systems such as the system shown in Fig. 1, the masses and the spring constants appear multilinearly in the numerator and denominator of the plant transfer function, as can be seen in Eq. (2). It is also shown in Ref. 13 that the ∞-norm parameter margin computation for a stabilized mass-spring system using a frequency-sweeping approach based on the mapping theorem can become numerically sensitive because of a possible discontinuity in frequency. Reference 14 analytically proves that the numerical sensitivity problem encountered in Ref. 13 is caused by actual discontinuities in frequency. As discussed in Refs. 14 and 15, the ∞-norm parameter margin of a controller designed for problem 2 can be simply found by checking for instability in the corner directions of the parameter space hypercube, at a finite number of critical frequencies.

In this section, the concept of internal feedback loop employed for linear systems described by first-order state-space equations is extended to structural dynamic systems described by second-order matrix differential equations with uncertain mass and stiffness matrices. The equations of motion of the system shown in Fig. 1 can be written as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}(w_1 + u) \tag{31}$$

where the three uncertain parameters m_1, m_2, and k appear linearly in the mass and stiffness matrices. For design simplicity, w_2 is assumed to be zero.

The perturbed parameters can be represented as

$$m_1 = 1 + \delta_1 \tag{32a}$$

$$m_2 = 1 + \delta_2 \tag{32b}$$

$$k = 1 + \delta_3 \tag{32c}$$

where δ_i represents the percentage perturbation from the nominal values of each parameter.

Substituting Eqs. (32) into Eq. (31), we get

$$\ddot{x}_1 = -(x_1 - x_2) - \delta_1\ddot{x}_1 - \delta_3(x_1 - x_2) + w_1 + u$$

$$\ddot{x}_2 = (x_1 - x_2) - \delta_2\ddot{x}_2 + \delta_3(x_1 - x_2) \tag{33}$$

Define the following new variables:

$$\tilde{z} \triangleq \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_3 \end{bmatrix} = \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ x_1 - x_2 \end{bmatrix} \tag{34a}$$

$$\tilde{w} \triangleq \begin{bmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_3 \end{bmatrix} = -E\tilde{z} \tag{34b}$$

where

$$E = \text{diag}\{\delta_1, \delta_2, \delta_3\} \tag{35}$$

Equation (33) can then be expressed as

$$\ddot{x}_1 = -(x_1 - x_2) + \tilde{w}_1 + \tilde{w}_3 + w_1 + u \tag{36a}$$

$$\ddot{x}_2 = (x_1 - x_2) + \tilde{w}_2 - \tilde{w}_3 \tag{36b}$$

By defining

$$\hat{w} \leftarrow \begin{bmatrix} \tilde{w} \\ w_1 \\ v \end{bmatrix}, \qquad \hat{z} \leftarrow \begin{bmatrix} \tilde{z} \\ x_2 \\ u \end{bmatrix} \tag{37}$$

we get the modified system matrices in Eq. (14) as

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \end{bmatrix} W, \qquad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$D_{11} = \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} W$$

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}, \qquad D_{22} = 0$$

$$D_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \end{bmatrix} W$$

where W is a diagonal weighting matrix for $\hat{w}$.

Since $D_{11} \neq 0$ here, the Glover-Doyle algorithm[3] is employed to obtain a robust H_∞-suboptimal controller for problem 2 with the three uncertain parameters m_1, m_2, and k.

After trial-and-error iterations, we select $\gamma = 1$ and the disturbance weighting matrix

$$W = \text{diag}\{0.001, 0.001, 0.044, 0.036, 0.224\} \tag{38}$$

to obtain the following controller:

$$\dot{\hat{x}}_c = A_c \hat{x}_c + Ly \tag{39a}$$

$$u = -Kx_c \tag{39b}$$

where

$$A_c = \begin{bmatrix} 0.125 & -0.288 & 1.059 & 0.008 \\ -0.112 & -0.553 & -0.052 & 0.993 \\ -2.575 & 1.808 & -2.149 & -0.277 \\ 1.007 & -1.197 & 0.004 & 0.001 \end{bmatrix}$$

$$L = \begin{bmatrix} 0.195 & 0.679 & -0.036 & 0.201 \end{bmatrix}^T$$

$$K = \begin{bmatrix} 1.572 & -0.772 & 2.145 & 0.277 \end{bmatrix}$$

The controller in transfer-function form is

$$u(s) = \frac{-0.0728(s/0.132+1)(s/1.075+1)}{[(s/0.857)^2 + 2(0.877)s/0.857 + 1]} \times \frac{(-s/3.876+1)}{[(s/1.573)^2 + 2(0.341)s/1.573 + 1]} y(s) \tag{40}$$

which is a nonminimum-phase compensator.

Figure 4 shows a closed-loop root locus vs overall loop gain. The nominal system has a 6.1 dB gain margin and a 34.13 deg phase margin. A significant improvement of gain and phase margins is achieved over the controller designed for problem 1 with a 3.35 dB gain margin and a 24.45 deg phase margin. Such an improvement is because the control design for problem 2 with three uncertain parameters also attempts to maximize the gain margin, since the parametric variation in the corner directions of $\delta_1 = \delta_2 = \delta_3$ corresponds to a pure loop-gain variation.

The nominal closed-loop poles are

$$-0.232 \pm 0.192j, \quad -0.459 \pm 0.394j$$
$$-0.172 \pm 1.431j, \quad -0.425 \pm 1.318j$$

The settling time of the nominal closed-loop system can be estimated as $4/0.232 = 17.24$ s, based on the time constant of the closed-loop rigid-body mode pole. Figure 5 shows that the nominal closed-loop system settles down within approximately 15 s for a unit impulse exerted on body 2.

As discussed, the ∞-norm parameter margin is employed as a stability robustness measure, and it is found by checking for instability in the corner directions of the parameter space hypercube. The ∞-norm parameter margin, denoted as p_m, of the preceding controller designed for problem 2 can be found as $p_m = 0.46$ with the critical frequency of 0.748 rad/s and the critical parameters of $m_1 = m_2 = 1.46$ and $k = 0.54$, as illustrated in Fig. 4. The critical corner corresponds to an increase in both the masses and a decrease in the spring stiffness.

A significant improvement of the parameter margin is also achieved over the controller designed for problem 1 with the parameter margin of $p_m = 0.32$ at the critical frequency of 0.643 rad/s. The critical parameters are $m_1 = m_2 = k = 0.68$, as illustrated in Fig. 2. The critical corner shown in Fig. 2 corresponds to decreasing all three parameters; thus it corresponds to a pure loop-gain increase. Consequently, a 0.32 parameter margin at this critical corner corresponds to a gain margin of 3.35 dB.

It seems that the control design for problem 2 has achieved the primary goal of finding a controller that yields the largest hypercube that fits within the stable domain in the plant's parameter space, subject to all other design requirements.

VI. Robust Control Design for Problem 3

In this section, the robust H_∞ control design methodology presented in Sec. III is incorporated with the internal model principle for robust asymptotic disturbance rejection and is applied to problem 3.

Consider a sinusoidal disturbance described by

$$w_1(t) = w_2(t) = \sin(0.5t) \tag{41}$$

which is acting on body 1 and/or body 2 of the two-mass-spring model in Fig. 1. It is assumed that the amplitude and phase of the disturbance are constant but unknown. To

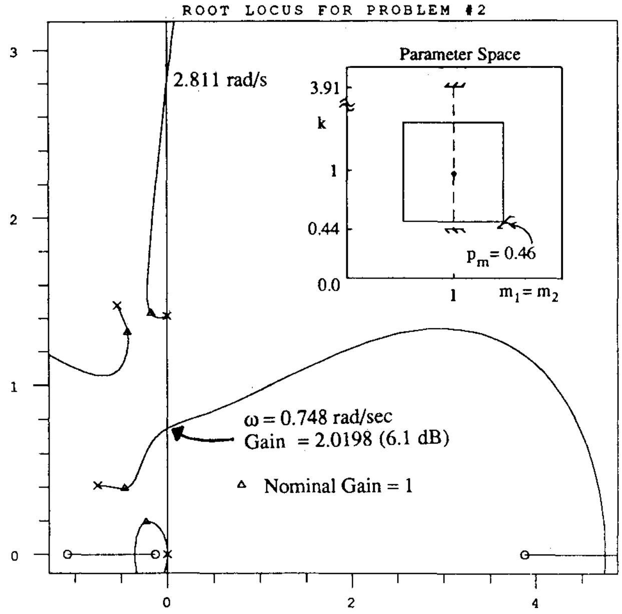

Fig. 4 Root-locus vs overall loop gain (problem 2).

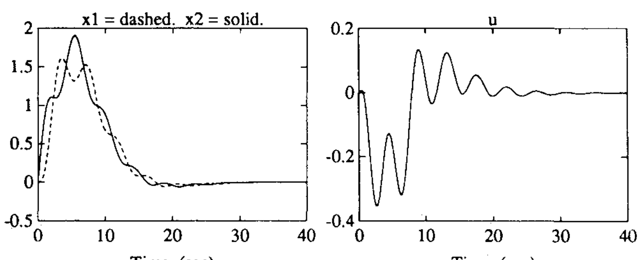

Fig. 5 Time responses to a unit-impulse disturbance acting on body 2 (problem 2).

achieve asymptotic disturbance rejection, an internal model of the disturbance, which is driven by the measured output, must be introduced in the feedback loop. In this approach, the actual location of external persistent disturbances is not important.

The disturbance rejection filter is assumed to have the form

$$\ddot{\alpha} + (0.5)^2\alpha = y \tag{42}$$

This model can be represented in state-space form as

$$\dot{x}_d = A_d x_d + B_d y \tag{43}$$

where

$$A_d = \begin{bmatrix} 0 & 1 \\ -0.25 & 0 \end{bmatrix}$$

$$B_d = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad x_d = \begin{bmatrix} \alpha \\ \dot{\alpha} \end{bmatrix}$$

The state-space equation for the two-mass-spring model, described by Eqs. (1) with $m_1 = m_2 = 1$, is augmented to include the internal disturbance model as follows:

$$\dot{x} = Ax + B_1 w + B_2 u$$

$$z = C_1 x + D_{11} w + D_{12} u$$

$$y = C_2 x + D_{21} w + D_{22} u \tag{44}$$

where $x = [x_p^T, x_d^T]^T$ is the augmented state vector, x_p is the plant state vector, $w = [w_1, v]^T$, and

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ -k & k & 0 & 0 & 0 & 0 \\ k & -k & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -0.25 & 0 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & w_\alpha & w_{\dot{\alpha}} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \qquad D_{21} = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

and $D_{11} = 0$, $D_{22} = 0$; w_α and $w_{\dot{\alpha}}$ are weighting factors for α and $\dot{\alpha}$, respectively.

The system matrix perturbation due to the uncertainty of k can be decomposed as

$$\Delta A = -M_x \Delta k N_x \tag{45}$$

where M_x and N_x span the columns and rows of ΔA, respectively, and are written as

$$M_x = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \qquad N_x = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

One approach for applying the robust H_∞ control theory, incorporating the internal model principle, is to directly follow the procedure described in the preceding section. In this case, the controller equation for the augmented system is expressed in state-space form as follows:

$$\begin{bmatrix} \dot{\hat{x}}_p \\ \dot{\hat{x}}_d \end{bmatrix} = A_c \begin{bmatrix} \hat{x}_p \\ \hat{x}_d \end{bmatrix} + Ly \tag{46a}$$

$$u = -K \begin{bmatrix} \hat{x}_p \\ \hat{x}_d \end{bmatrix} \tag{46b}$$

where

$$A_c = A + \gamma^{-2} B_1 B_1^T X - B_2 K - L C_2 \tag{47}$$

and K, L, and X are obtained by solving two Riccati equations, Eqs. (18) and (19), for the modified, augmented system described by Eq. (44). This approach, however, fails to provide a compensator with poles exactly at the disturbance frequency (i.e., at $\pm 0.5j$), since the disturbance filter model is driven by the estimated state $\hat{x}_2$, not directly by the measured x_2. As a result, the effect of the sinusoidal disturbance, though reduced significantly, cannot be completely rejected using this approach.

To have $\pm 0.5j$ as compensator poles, the following simple modification is proposed here without a formal mathematical proof. However, as can be seen later in this section, the proposed approach does not significantly change the overall closed-loop stability and performance of a controller obtained using Eqs. (46).

First, the original controller equation given by Eqs. (46) for the augmented system is rewritten as

$$\begin{bmatrix} \dot{\hat{x}}_p \\ \dot{\hat{x}}_d \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ \hat{x}_d \end{bmatrix} + \begin{bmatrix} L_p \\ L_d \end{bmatrix} y \tag{48a}$$

$$u = -\begin{bmatrix} K_p & K_d \end{bmatrix} \begin{bmatrix} \hat{x}_p \\ \hat{x}_d \end{bmatrix} \tag{48b}$$

where A_{ij} are the submatrices of A_c.

Then, $\hat{x}_d$ is simply replaced by x_d with the following substitutions in Eqs. (48):

$$x_d \leftarrow \hat{x}_d, \qquad B_d \leftarrow L_d$$

$$0 \leftarrow A_{21}, \qquad A_d \leftarrow A_{22} \tag{49}$$

Finally, the robust H_∞ controller incorporating the internal model principle is expressed in state-space form as

$$\begin{bmatrix} \dot{\hat{x}}_p \\ \dot{x}_d \end{bmatrix} = A_c \begin{bmatrix} \hat{x}_p \\ x_d \end{bmatrix} + B_c y \tag{50a}$$

$$u = -C_c \begin{bmatrix} \hat{x}_p \\ x_d \end{bmatrix} \tag{50b}$$

where

$$A_c = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_d \end{bmatrix}$$

$$B_c = \begin{bmatrix} L_p \\ B_d \end{bmatrix}$$

$$C_c = \begin{bmatrix} K_p & K_d \end{bmatrix}$$

After a certain amount of trial and error, we select $\gamma = 10$, $w_\alpha = 0$, $w_{\dot{\alpha}} = 0.35$. The disturbances $\tilde{w}$, w_1, and v are weighted by 0.2, 0.018, and 0.13, respectively. The controller corresponding to these weighting factors can be found as

$$A_c = \begin{bmatrix} 0 & 0.46 & 1 & 0 & 0 & 0 \\ 0 & -1.36 & 0 & 1 & 0 & 0 \\ -2.97 & 2.07 & -1.98 & -1.57 & 0.17 & 0.09 \\ 1 & -1.92 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -0.25 & 0 \end{bmatrix}$$

$$B_c = \begin{bmatrix} -0.46 & 1.36 & -0.79 & 0.92 & 0 & 1 \end{bmatrix}^T$$

$$C_c = \begin{bmatrix} 1.97 & -0.29 & 1.98 & 1.57 & -0.17 & 0.09 \end{bmatrix}$$

or in transfer function form as

$$u(s) = \frac{-0.0286(s/0.08+1)(-s/0.81+1)(s/1.74+1)}{\left[(s/1.17)^2+2(0.86)s/1.17+1\right]} \times \frac{\left[(s/0.47)^2-2(0.18)s/0.47+1\right]}{\left[(s/1.88)^2+2(0.35)s/1.88+1\right]\left[(s/0.5)^2+1\right]} y(s) \quad (51)$$

which is a sixth-order nonminimum-phase compensator. It can be seen that for asymptotic disturbance rejection, the compensator has poles at $\pm 0.5j$ with the associated nonminimum-phase zeros at $+0.087 \pm 0.463j$. Such a pole-zero pair is called a disturbance rejection "dipole" in Ref. 10.

Figure 6 shows a closed-loop root locus vs overall loop gain. The nominal system has a 4.34 dB gain margin and a 28.6 deg phase margin. (Note that, similar to problem 1, loop gain uncertainty is not considered in control design for problem 3.)

The closed-loop poles of the nominal system are

$$-0.167 \pm 0.164j, \quad -0.168 \pm 0.494j$$
$$-0.467 \pm 0.419j, \quad -0.358 \pm 1.485j$$
$$-0.511 \pm 1.503j$$

The settling time of the nominal closed-loop system can be estimated as $4/0.168 = 23.8$ s, based on the time constant of the closed-loop pole of the disturbance rejection filter.

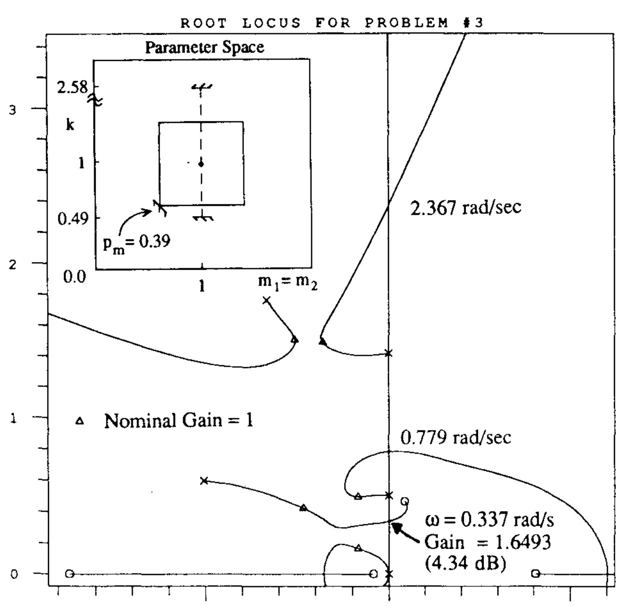

Fig. 6 Root-locus vs overall loop gain (problem 3).

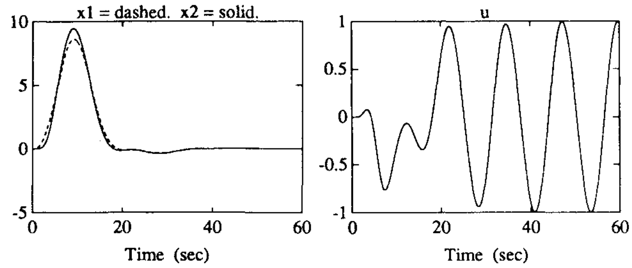

Fig. 7 Time responses for problem 3 with $w_1(t) = \sin 0.5t$ and $w_2(t) = 0$.

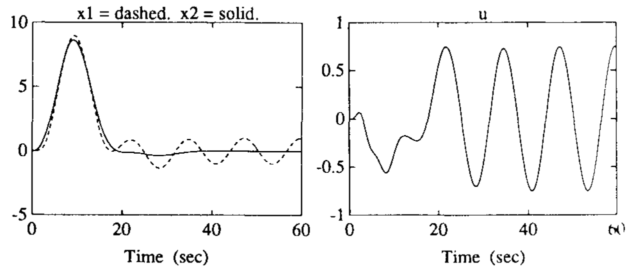

Fig. 8 Time responses for problem 3 with $w_1(t) = 0$ and $w_2(t) = \sin 0.5t$.

The time responses to $w_1(t) = \sin(0.5t)$ and $w_2(t) = \sin(0.5t)$ are shown in Figs. 7 and 8, respectively. It can be seen that the controlled output x_2 settles down within 20 s, and robust asymptotic disturbance rejection is achieved because of the compensator poles exactly at $\pm 0.5j$.

The closed-loop system is stable for $0.49 \leq k \leq 2.58$, which corresponds to the parameter perturbation range of

$$-0.51 \leq \Delta k \leq +1.58$$

If the closed-loop system has three uncertain parameters m_1, m_2, and k similar to problem 2, the ∞-norm parameter margin can be found as $p_m = 0.394$ with the critical frequency of 0.337 rad/s and the critical parameters of $m_1 = m_2 = k = 0.606$, as illustrated in Fig. 6.

VII. Discussion

A few comments on the merits and drawbacks of nonminimum-phase compensation are made here. In particular, an issue regarding the relatively small gain and phase margins of nonminimum-phase compensation in general, compared with typical 6 dB and 40 deg margins employed in practice, is briefly discussed here.

For the benchmark problem, it is possible to design a minimum-phase compensator with relatively large phase margin (e.g., see Refs. 16 and 17). However, such a high-pass, phase-lead, minimum-phase controller (which is basically a "differentiating" compensator) inevitably results in a large loop-gain increase at high frequencies (e.g., a 100 dB increase[17]). It would amplify any measurement noise intolerably and often result in actuator saturation. Furthermore, it may destabilize any unmodeled high-frequency flexible modes because of its large gain increase at high frequencies.

On the other hand, most nonminimum-phase compensation results in significant rolloffs, at the expense of relatively small gain/phase margins at low frequencies. Consequently, such relatively small gain/phase margins are often inevitable for a nonminimum-phase compensator that is properly designed for structured parameter uncertainty. It is, however, possible to improve gain/phase margins of a nonminimum-phase compensator by including the loop gain and/or phase uncertainty as one of structured uncertain parameters.

In problem 2 with three uncertain parameters, the control design also attempts to maximize the gain margin, since parameter variation in the corner directions of $\delta_1 = \delta_2 = \delta_3$ corresponds to pure loop-gain variation. Consequently, the control design for problem 2 has resulted in a 6.1 dB gain margin and a 34.13 deg phase margin, which is a significant improvement over the control design for problem 1 in which loop-gain uncertainty was not considered in design.

VIII. Conclusions

A robust H_∞ control synthesis technique and its application to a benchmark problem have been presented. The proposed design methodology also incorporates the internal model principle for robust asymptotic disturbance rejection in the presence of persistent external disturbances. The ∞-norm parameter margin as well as the conventional gain/phase margins was employed as a stability robustness measure of a closed-loop system with structured multiparameter variations.

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References

[1]Wie, B., and Bernstein, D., "A Benchmark Problem for Robust Control Design," *Proceedings of the 1990 American Control Conference*, May 23–25, 1990, pp. 961–962; also *Proceedings of the 1991 American Control Conference*, June 26–28, 1991, pp. 1929–1930.

[2] Bryson, A. E., Jr., "Some Connections Between Modern and Classical Control Concepts," *ASME Journal of Dynamic Systems, Measurement, and Control*, Vol. 101, June 1979, pp. 91–98.

[3]Glover, K., and Doyle, J., "State-Space Formulae for All Stabilizing Controllers That Satisfy an H_∞-Norm Bound and Relations to Risk Sensitivity," *System & Control Letters*, Vol. 11, 1988, pp. 167–172.

[4]Doyle, J., Glover, K., Khargonekar, P., and Francis, B., "State-Space Solutions to Standard H_2 and H_∞ Control Problems," *IEEE Transactions on Automatic Control*, Vol. 34, No. 8, Aug. 1989, pp. 831–847.

[5]Byun, K.-W., "Robust Control Synthesis for Uncertain Dynamical Systems," Ph.D. Dissertation, Dept. of Aerospace Engineering, Univ. of Texas at Austin, Austin, TX, May 1990.

[6]Byun, K.-W., Wie, B., and Sunkel, J., "Robust Non-Minimum-Phase Compensation for a Class of Uncertain Dynamical Systems," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 6, 1991, pp. 1191–1199.

[7]Byun, K.-W., Wie, B., Geller, D., and Sunkel, J., "Robustified H_∞ Control Design for the Space Station with Structured Parameter Uncertainty," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 6, 1991, pp. 1115–1122.

[8]Chen, C.T., *Linear System Theory and Design*, CBS College Publishing, New York, 1984, pp. 488–506.

[9]Wie, B., Byun, K.-W., Warren, W., Geller, D., Long, D., and Sunkel, J., "New Approach to Attitude and Momentum Control for the Space Station," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 5, 1989, pp. 714–722.

[10]Wie, B., and Gonzalez, M., "Control Synthesis for Flexible Space Structures Excited by Persistent Disturbances," *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 1, 1992, pp. 73–80.

[11]Wie, B., Horta, L., and Sulla, J., "Classical Control System Design and Experiment for the Mini-Mast Truss Structure," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 4, 1991, pp. 778–784.

[12]Wie, B., and Byun, K.-W., "New Generalized Structural Filtering Concept for Active Vibration Control Synthesis," *Journal of Guidance, Control, and Dynamics*, Vol. 12, No. 2, 1989, pp. 147–154.

[13]Wedell, E., Chuang, C.-H., and Wie, B., "Stability Robustness Margin Computation for Structured Real-Parameter Perturbations," *Journal of Guidance, Control, and Dynamics*, Vol. 14, No. 3, 1991, pp. 607–614.

[14]El Ghaoui, L., and Bryson, A. E., "Worst-Case Parameter Change for Stabilized Conservative SISO Systems," *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, AIAA, Washington, DC, 1991, pp. 1490–1495.

[15]Warren, W., and Wie, B., "Parameter Margins for Stabilized Conservative Multilinear Systems," *Proceedings of the 1991 American Control Conference*, Boston, MA, June 26–28, 1991, pp. 1933–1934.

[16]Franklin, G. F., and Powell, J. D., *Feedback Control of Dynamic Systems*, Addison-Wesley, Reading, MA, 1986, pp. 450–473.

[17]Chiang, R. Y., and Safonov, M. G., "H_∞ Robust Control Synthesis for an Undamped, Noncolocated Spring-Mass System," *Proceedings of 1990 American Control Conference*, May 1990, pp. 966–967.